PHYSICAL REVIEW D, VOLUME 65, 104037

Vacuum states and the *S* matrix in dS/CFT

Marcus Spradlin*
Department of Physics, Princeton University, Princeton, New Jersey 08544

Anastasia Volovich†
Department of Physics, Harvard University, Cambridge, Massachusetts 02138

(Received 7 January 2002; published 15 May 2002)

We propose a definition of dS/CFT correlation functions by equating them to *S*-matrix elements for scattering particles from $\mathcal{I}^-$ to $\mathcal{I}^+$. In planar coordinates, which cover half of de Sitter space, we consider instead the *S* vector obtained by specifying a fixed state on the horizon. We construct the one-parameter family of de Sitter invariant vacuum states for a massive scalar field in these coordinates, and show that the vacuum obtained by analytic continuation from the sphere has no particles on the past horizon. We use this formalism to provide evidence that the one-parameter family of vacua corresponds to marginal deformations of the CFT by computing a three-point function.

DOI: 10.1103/PhysRevD.65.104037 PACS number(s): 04.62.+v

I. INTRODUCTION

Understanding quantum gravity in de Sitter space remains one of the most important problems in theoretical physics. A correspondence relating gravity in de Sitter space to a conformal field theory (CFT) has recently been suggested [1] (see also [2]), and subsequently studied by several authors [3]. Recent works on de Sitter space include [4].

The dS/CFT correspondence is modeled in analogy with the AdS/CFT correspondence [5,6,7], which has proven to be phenomenally successful. But it is important to keep in mind that in the prehistoric days of AdS/CFT, when the first signs were emerging that there might be some connection between supergravity on AdS space and conformal field theories, it would have seemed beyond hope to expect that these developments would lead to a nonperturbative definition of quantum gravity on AdS space, and to all of the remarkable advances that have been made in our understanding of gauge theories. AdS/CFT turned out to be more wonderful than we had any right to expect, so we should not be prejudiced against dS/CFT simply because it has some mysterious and confusing aspects and has not yet borne the rich fruit of its AdS brother.

Therefore we proceed modestly in this paper, by elucidating the connection between gravity on de Sitter space and conformal field theory correlation functions. Our probe will be an interacting real scalar field of mass m. This turns out to be more interesting than it might seem at first since it is known that there is no unique de Sitter invariant vacuum state for a massive scalar field, but instead a family of vacua labeled by a complex parameter γ. Changing the vacuum $|\gamma\rangle$ in the bulk of dS_3 has been argued to correspond to a marginal deformation of the associated CFT [8].

The central result of this paper is a proposal for how to extract CFT correlation functions from n-point correlation functions of the scalar field on dS_3. Along the way we highlight the important differences between dS and AdS which make naive extrapolation of some AdS/CFT results problematic. In the global picture of de Sitter, there are four CFT operators associated with the scalar field ϕ, which are labeled $\mathcal{O}_{\pm}^{\text{in,out}}$, and have weights $h_{\pm}=1\pm\sqrt{1-m^2}$. Only two of these operators are independent, and in general the out operators can be related to the in operators by path integral evolution from $\mathcal{I}^-$ to $\mathcal{I}^+$. We equate correlation functions of $\mathcal{O}_{+}^{\text{in,out}}$ with *S*-matrix elements for particles coming in from $\mathcal{I}^-$ and going out to $\mathcal{I}^+$.[1] This definition of CFT correlation functions is motivated by a similar construction for AdS which has been developed in [10–14]. We do not address the important issue that these *S*-matrix elements are only "metaobservables" and cannot be probed by any single observer in de Sitter space [15,16,2].

In planar coordinates, which only cover half of de Sitter space, one has only half as many operators. For example, in the patch $\mathcal{O}^-$ which includes the causal past of an observer sitting at the south pole, there are no asymptotic out states, so the best one can do is to study the *S* vector [2,15]. This leads to a natural definition of correlation functions involving only $\mathcal{O}_{\pm}^{\text{in}}$. Along the way, we prove the somewhat surprising result that the Euclidean vacuum state (which is the one obtained by analytic continuation from the sphere to de Sitter spacetime) is the state with no particles on the horizon.

The plan of the paper is the following. In Sec. II we introduce global and planar coordinate systems for dS_3, mode expansions for the scalar field, and the bulk-boundary propagators. In Sec. III we review the construction of the de Sitter invariant vacuum states $|\gamma\rangle$ in global coordinates and record the two-point functions of the scalar field. In Sec. IV we show how these vacuum states can be obtained naturally in planar coordinates as well, and that the Euclidean vacuum is the one with no particles on the horizon. Section V con-

[1] Our *S* matrix is the standard one of perturbative quantum field theory, as distinct from the (finite-dimensional) matrices of [2,9], although it would be very interesting to understand a connection with these works.

*Email address: spradlin@feynman.princeton.edu
†Email address: nastya@gauss.harvard.edu

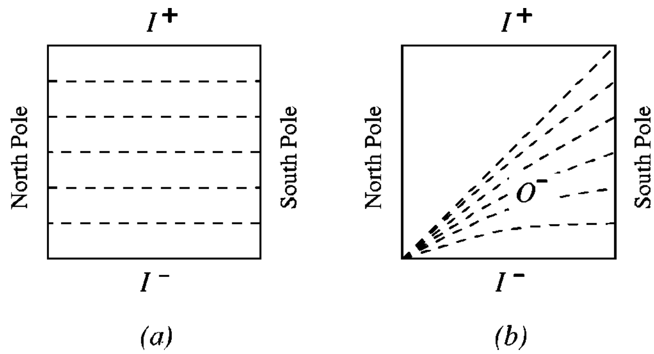

FIG. 1. The Penrose diagram for de Sitter space. (a) Global coordinates cover all of de Sitter space, with dotted lines signifying slices of constant τ, which ranges from $-\infty$ to $+\infty$. (b) Planar coordinates cover only the causal past $\mathcal{O}^-$ of an observer at the south pole. The dotted lines are lines of constant t, with $t=0$ on $\mathcal{I}^-$ and $t\to+\infty$ at the horizon.

tains the general prescription for calculating CFT correlation functions in global coordinates and explains the connection to the S matrix. At the end of Sec. V we outline the calculation of a CFT three-point function and show that the parameter γ appears nontrivially in an invariant ratio of correlation functions, providing an evidence that these vacua are marginal deformations of the associated CFT. The prescription for dS/CFT correlation functions in planar coordinates appears in Sec. VI, where the motivation is provided by the S vector.

II. COORDINATES, MODES AND BULK-BOUNDARY PROPAGATORS

In this paper we consider an interacting scalar field ϕ in dS_3 with the action

$$S=-\frac{1}{2}\int\sqrt{-g}[(\nabla\phi)^2+m^2\phi^2+V(\phi)]. \qquad (2.1)$$

We set the de Sitter radius l to unity and assume that $m^2>1$. This condition, while not essential, simplifies the discussion for reasons that will become clear shortly. Most of the results of this paper generalize more or less straightforwardly to scalars with $m^2\leqslant 1$, higher spin fields, and higher dimensional de Sitter space. We will comment on exceptions to this expectation as they arise.

We consider two coordinate systems: global coordinates (τ,Ω) and planar coordinates $(t,\vec{x})$ (see Fig. 1). Here Ω is a point on S^2 and $\vec{x}$ is a point on $\mathbf{R}^2$. The metric is

$$ds^2=-d\tau^2+\cosh^2\tau d\Omega_2^2=\frac{1}{t^2}(-dt^2+d\vec{x}^2). \qquad (2.2)$$

Global coordinates cover all of dS_3, with τ running from $-\infty$ on $\mathcal{I}^-$ to $+\infty$ on $\mathcal{I}^+$, while planar coordinates only cover the causal past of an observer on the south pole. In planar coordinates $\mathcal{I}^-$ is at $t=0$ and the horizon lies at $t=+\infty$. A number of additional coordinate systems and further details can be found in [17].

For some purposes, particularly cosmology, the region $\mathcal{O}^+$ corresponding to the causal future of an observer at the south pole may be of greater interest than $\mathcal{O}^-$ (see [18] for an interesting application). All of the formulas presented in this paper can be adapted to $\mathcal{O}^+$ by taking $t\to -t$, so that $t=0$ corresponds to $\mathcal{I}^+$ and $t=-\infty$ corresponds to the horizon.

In global coordinates we will make use of the antipodal map on dS_3. Actually there are two antipodal maps, one which just takes Ω to the antipodal point on S^2, and one which in addition takes $\tau\to-\tau$. We will use the notation Ω_{A} for the former and x_{A} for the latter, where $x=(\tau,\Omega)$.

The following two subsections catalog the mode expansions for a free scalar field and introduce the bulk-boundary propagators in the two coordinate systems.

A. Global coordinates

In global coordinates we follow closely the conventions of [8]. A basis of positive frequency solutions of the free Klein-Gordon equation is given by

$$\phi_{lm}(\tau,\Omega)=y_l(\tau)Y_{lm}(\Omega), \qquad (2.3)$$

where

$$y_l(\tau)=e^{i\theta_l}\sqrt{\frac{2}{\mu}}(1+e^{2\tau})^l e^{(1-i\mu)\tau} \times F(l+1,l+1-i\mu,1-i\mu,-e^{2\tau}), \qquad (2.4)$$

and the phase[2] θ_l is defined by

$$e^{2i\theta_l}=(-1)^{l+1}\frac{\Gamma(i\mu)\Gamma(l+1-i\mu)}{\Gamma(-i\mu)\Gamma(l+1+i\mu)}. \qquad (2.5)$$

The quantity $\mu\equiv\sqrt{m^2-1}$ must be real in order for ϕ_{lm} and ϕ^*_{lm} to be interpreted in the usual way as positive and negative frequency modes, respectively.[3] Note that as in [8] we find it convenient to use a nonstandard basis of spherical harmonics. We define

$$Y_{lm}=\sqrt{\frac{i}{2}}S_{lm}+(-1)^l\sqrt{-\frac{i}{2}}S^*_{lm} \qquad (2.6)$$

in terms of the usual spherical harmonics S_{lm}, their utility for our purpose being that they satisfy

$$Y^*_{lm}(\Omega)=(-1)^l Y_{lm}(\Omega)=Y_{lm}(\Omega_A). \qquad (2.7)$$

The modes (2.3) are normalized with respect to the Klein-Gordon inner product

[2]The reader may well wonder why we have bothered to introduce such a complicated phase, since the overall phase of a mode function is of course irrelevant. It turns out that the definition (2.4) will ultimately prove to be very convenient because $y_l(-\tau)=y_l(\tau)^*$.

[3]The analysis still goes through for $0<m^2<1$, although the case $m^2=0$ is quite subtle [19] and will not be considered here.

$$\langle \phi_{lm},\phi_{l'm'}\rangle = i(\cosh\tau)^2 \int d^2\Omega(\phi^*_{lm}\overleftrightarrow{\partial}_\tau \phi_{l'm'}) = \delta_{ll'}\delta_{mm'} \tag{2.8}$$

by virtue of the fact that

$$i(\cosh\tau)^2(y_l^*\overleftrightarrow{\partial}_\tau y_l) = 1 \tag{2.9}$$

for all l.

The phase $e^{2i\theta_l}$ will play an important role below, so we record some of its properties. We define

$$\Delta_\pm(\Omega,\Omega') = -\frac{1}{\mu \sinh \pi\mu}\sum_{lm} Y_{lm}(\Omega)Y_{lm}(\Omega')e^{\mp 2i\theta_l}, \tag{2.10}$$

which is just the two point function for a conformal field of dimension $h_\pm \equiv 1\pm i\mu$ on the sphere [8]. It is clear that they satisfy

$$(\mu\sinh\pi\mu)^2\int d^2\Omega''\Delta_-(\Omega,\Omega'')\Delta_+(\Omega'',\Omega') = \delta^2(\Omega,\Omega'). \tag{2.11}$$

Next we discuss the bulk-boundary propagators,[4] which are used to construct bulk solutions of the Klein-Gordon equation corresponding to wave packets coming in from $\mathcal{I}^-$ or going out to $\mathcal{I}^+$. We define $K^\pm$ by

$$K^\pm(\Omega';\tau,\Omega) = \sum_{lm} Y_{lm}(\Omega')K^\pm_{lm}(\tau,\Omega),$$
$$K^\pm_{lm}(\tau,\Omega) = e^{\pm i\theta_l}\sqrt{\frac{\mu}{2}}Y^*_{lm}(\Omega)y_l(\tau). \tag{2.12}$$

These are related by

$$K^\pm(\Omega';\tau,\Omega) = -\mu\sinh\pi\mu\int d^2\Omega''\Omega_\mp(\Omega',\Omega'') \times K^\mp(\Omega''_A;\tau,\Omega) \tag{2.13}$$

and satisfy

$$K^\pm(\Omega';\tau,\Omega) = K^\pm(\Omega;\tau,\Omega') = K^\pm(\Omega'_A;\tau,\Omega_A). \tag{2.14}$$

They are solutions of the wave equation with the boundary conditions

$$\lim_{\tau\to\pm\infty} K^\pm(\Omega',\tau,\Omega) = e^{(\mp 1 - i\mu)\tau}\delta^2(\Omega,\Omega') + \mathcal{O}(e^{\mp 3\tau}), \tag{2.15}$$

[4]Although de Sitter space itself has no boundary, $\mathcal{I}^+$ and $\mathcal{I}^-$ are the boundaries of the conformal compactification of de Sitter space.

i.e., they are positive frequency solutions which approach delta functions on $\mathcal{I}^\pm$. For this reason we will frequently use the notation $K^{\rm in}\equiv K^-$ and $K^{\rm out}\equiv K^+$. Given any smooth function $f(\Omega)$ on the sphere, we can construct solutions of the bulk Klein-Gordon equation by the prescription

$$\phi_f^{\rm in,out}(\tau,\Omega) = \int d^2\Omega' f(\Omega')K^{\rm in,out}(\Omega';\tau,\Omega). \tag{2.16}$$

The solution $\phi_f^{\rm in}$ represents a wave packet with envelope f coming in from $\mathcal{I}^-$, while $\phi_f^{\rm out}$ represents a wave packet with envelope f going out to $\mathcal{I}^+$.

B. Planar coordinates

A basis of positive frequency solutions of the free Klein-Gordon equation is given by

$$\phi_{\vec p}(t,\vec x) = e^{i\vec p\cdot\vec x}u(p,t), \quad u(p,t) = \frac{tJ_-(pt)}{\sqrt{8\pi\sinh\pi\mu}}. \tag{2.17}$$

We use the notation $J_\pm(z)\equiv J_{\pm i\mu}(z)$, where $J_\nu(z)$ is the Bessel function. Thus $\phi(t)\sim t^{1-i\mu}$ near $t=0$. The modes (2.17) are normalized according to

$$\langle\phi_{\vec p},\phi_{\vec p'}\rangle = \frac{i}{t}\int d^2x(\phi^*_{\vec p}\overleftrightarrow{\partial}_t\phi_{\vec p'}) = \delta^2(\vec p-\vec p'). \tag{2.18}$$

The bulk-boundary propagator to $\mathcal{I}^-$ is

$$K(\vec y;t,\vec x) = \frac{1}{2\pi}\int d^2p\, e^{i\vec p\cdot\vec y}\tilde K(\vec p;t,\vec x),$$
$$\tilde K(\vec p;t,\vec x) = e^{-i\vec p\cdot\vec x}z(p)u(p,t), \tag{2.19}$$

where

$$z(p) = \frac{1}{2\pi}(p/2)^{i\mu}\Gamma(1-i\mu)\sqrt{8\pi\sinh\pi\mu}. \tag{2.20}$$

This factor will play as important a role as the phase (2.5) in global coordinates. Performing the Fourier transform (2.19) gives

$$K(\vec y;t,\vec x) = -\frac{i\mu}{\pi}\theta(t-|\vec x-\vec y|)\left(\frac{t}{t^2-|\vec x-\vec y|^2}\right)^{1+i\mu}. \tag{2.21}$$

The solution to the free Klein-Gordon equation corresponding to an incoming wave packet with profile $f(\vec y)$ from $\mathcal{I}^-$ is then just

$$\phi_f(t,\vec x) = \int d^2y\, f(\vec y)K(\vec y;t,\vec x). \tag{2.22}$$

III. VACUUM STATES IN de SITTER SPACE

It was shown by Breitenlohner and Freedman [20] that in AdS_{d+1}, a scalar field whose mass lies in the range $-(d/2)^2<m^2<-(d/2)^2+1$ admits two inequivalent quantizations. Such scalars were later found to play an interesting role in the AdS/CFT correspondence [21,22]. Since this phenomenon is related to the fact that for this range of m^2, both mode solutions of the Klein-Gordon equation are normalizable [23], one might expect a similarly interesting story in the dS/CFT correspondence, where both modes are normalizable for any value of m^2.

In fact, as discussed in [24–27] and reviewed in [8], the story is even richer for de Sitter: there is a one complex parameter family of $SO(d,1)$ invariant vacuum states for a massive scalar field in d-dimensional de Sitter spacetime (of which a one real parameter subset is *CPT* invariant [8]). Two vacuum states play special roles: the Euclidean vacuum $|E\rangle$ is the one obtained by analytically continuing from the sphere to de Sitter spacetime, and the $|\mathrm{in}\rangle$ vacuum is the one with no particles on $\mathcal{I}^-$.[5]

It is occasionally said that the hypothesized dS/CFT correspondence is "just" an analytic continuation of AdS/CFT. However, it is easy to see that analytic continuation from AdS does not give any vacuum state for a scalar field in dS. Consider the AdS commutator function $[\Phi(x),\Phi(y)]$. It vanishes outside the light cone in AdS, but analytic continuation to dS involves interchanging the roles of t and r,[6] which turns light cones on their sides. So simply analytically continuing the two-point function for a scalar field would give a commutator function which vanishes inside the light cone, but not outside. This would violate causal propagation.

A. The MA transform

The vacuum associated to the global coordinate modes (2.3)[7] is called the $|\mathrm{in}\rangle$ vacuum since it corresponds to having no particles coming in from $\mathcal{I}^-$. Now consider the frequency independent (i.e., diagonal) Bogolyubov transformation

$$\tilde{\phi}_{lm}=\frac{1}{\sqrt{1-e^{\gamma+\gamma^*}}}(\phi_{lm}-e^{\gamma}\phi^*_{lm}). \tag{3.1}$$

Following [8], we call Eq. (3.1) an MA transform for Mottola and Allen [27,26]. The modes (3.1) define a de Sitter invariant vacuum state $|\gamma\rangle$ for any complex γ with $\mathrm{Re}(\gamma)<0$.[8] The two-point function $\langle\gamma|\Phi(x)\Phi(y)|\gamma\rangle$ has the usual singularity whenever x and y are null separated, but in general it has an additional singularity whenever x is null separated from y_A. Since this second pole is always separated from x by a horizon, there is no obvious reason to discard these vacuum states. The value $\gamma=-\pi\mu$ is the Euclidean vacuum, and $\gamma=-\infty$ is the $|\mathrm{in}\rangle$ vacuum. The two-point function in the Euclidean vacuum has no antipodal singularity.

B. Two-point functions in the $|\gamma\rangle$ vacua

In this section we record the two-point function $\langle\gamma|\Phi(x)\Phi(y)|\gamma\rangle$ in the $|\gamma\rangle$ vacuum for later use. It is straightforward to derive a general identity for the Wightman two-point function [8]

$$G^{\mathrm{W}}_{\gamma}(x,y)=\frac{1}{1-e^{\gamma+\gamma^*}}[G^{\mathrm{W}}_{\mathrm{in}}(x,y)-e^{\gamma^*}G^{\mathrm{W}}_{\mathrm{in}}(x,y_{\mathrm{A}}) + e^{\gamma+\gamma^*}G^{\mathrm{W}}_{\mathrm{in}}(y,x)-e^{\gamma}G^{\mathrm{W}}_{\mathrm{in}}(x_{\mathrm{A}},y)] \tag{3.2}$$

in terms of the Wightman function in the $|\mathrm{in}\rangle$ vacuum. It will be convenient to write an explicit formula, expressed in terms of the de Sitter invariant quantity P associated with two points [17]. In global coordinates,

$$P(\tau,\Omega;\tau',\Omega')=\cosh\tau\cosh\tau'\cos\Theta(\Omega,\Omega') -\sinh\tau\sinh\tau', \tag{3.3}$$

where Θ is the angle between Ω and Ω' on S^2, while in planar coordinates

$$P(t,\vec{x};t',\vec{x}')=1+\frac{(t-t')^2-|\vec{x}-\vec{x}'|^2}{2tt'}. \tag{3.4}$$

It is useful to keep in mind the following properties: $P(x,y)$ is greater than 1, equal to 1, or less than 1 respectively if x and y are timelike, null, or spacelike separated. Furthermore, $P(x,y)=-P(x,y_{\mathrm{A}})$ so that $P(x,y)$ is greater than -1, equal to -1, or less than -1 respectively if x and y_{A} are spacelike, null, or timelike separated.

In terms of the de Sitter invariant quantity P we can write the commutator function

$$iG^{\mathrm{C}}(x,y)\equiv[\Phi(x),\Phi(y)] =-\frac{i}{2\omega}\mathrm{sgn}(x^0-y^0)\frac{\cos[\mu\cosh^{-1}(P)]}{\sinh[\cosh^{-1}(P)]},\quad P>1. \tag{3.5}$$

Here $\mathrm{sgn}(x^0-y^0)$ is $+1$ if x is in the future light cone of y, and -1 if x is in the past light cone of y. Of course G^{C} vanishes for spacelike separation, $P<1$.

The commutator function is a c-number which is independent of the state $|\gamma\rangle$ [28], so we can summarize the γ dependence of the two-point function by looking at the Hadamard function, which turns out to be

---

[5]It was shown in [8] that in odd dimensional de Sitter spacetime, $|\mathrm{in}\rangle=|\mathrm{out}\rangle$, the state with no particles on $\mathcal{I}^+$.

[6]AdS and dS can both be obtained from Euclidean AdS with metric $(1/x_0^2)(dx_0^2+\cdots+dx_d^2)$, the only difference being whether one takes $(x_0,x_1)\to(it,r)$ or (r,it).

[7]By this we mean the vacuum annihilated by the operators multiplying Eq. (2.3) in the free field expansion of Φ.

[8]If $\mathrm{Re}(\gamma)>0$ then we can simply exchange ϕ and ϕ^*.

$$G^{\rm H}_{\gamma}(x,y)\equiv\langle\gamma|\{\Phi(x),\Phi(y)\}|\gamma\rangle$$
$$=-\frac{1}{\pi}\frac{1}{1-e^{\gamma+\gamma^*}}\frac{{\rm Im}\exp[\gamma-i\mu\cosh^{-1}(-P)]}{\sinh[\cosh^{-1}(-P)]},\quad P<-1,$$
$$=\frac{\cosh[\mu\cos^{-1}(P)]-\cosh[{\rm Re}(\gamma)]\cosh[\mu(\pi-\cos^{-1}(P))]}{2\pi\sinh\pi\mu\sinh[{\rm Re}(\gamma)]},\quad -1<P<1,\tag{3.6}$$
$$=\frac{\coth[{\rm Re}(\gamma)]}{2\pi}\frac{\sinh[\mu\cosh^{-1}(P)]}{\sinh[\cosh^{-1}(P)]},\quad P>1.$$

Note that only the $P<-1$ part is sensitive to the imaginary part of γ. The time-ordered correlation function

$$G^{\rm F}(x,y)_{\gamma}\equiv\langle\gamma|T\Phi(x)\Phi(y)|\gamma\rangle$$
$$=\sum_{lm}\theta(\tau-\tau')\tilde{\phi}_{lm}(x)\tilde{\phi}^*_{lm}(y)$$
$$+\theta(\tau'-\tau)\tilde{\phi}_{lm}(y)\tilde{\phi}^*_{lm}(x)\tag{3.7}$$

will play the central role beginning in Sec. V, and the representation (3.7) will prove more useful than the expression obtained after the sum is performed.

IV. THE MA′ TRANSFORM IN PLANAR COORDINATES

Previous analyses of these vacua have focused on global coordinates, where the calculations are simpler but the physical meaning of the Euclidean vacuum is obscure. In planar coordinates there is a new natural vacuum state: the one defined by having no particles on the horizon at $t=\infty$. However, the planar coordinate system only makes a subgroup of the full de Sitter isometry group manifest. In particular, the location of the horizon at $t=\infty$ is not invariant under de Sitter transformations, so one might have expected that the boundary condition of having no particles on the horizon could not give rise to a de Sitter invariant vacuum state. In this section we prove the slightly surprising result that the one parameter family of vacua do appear naturally in planar coordinates, and that the state with no particles on the horizon is just the Euclidean vacuum.

The MA transformation (3.1) cannot be done on the planar modes (2.17) since the resulting vacuum state would break translation invariance along $\vec{x}$—the cross terms between ϕ and ϕ^* would give rise to terms in the two point function depending on $|\vec{x}+\vec{y}|$ instead of $|\vec{x}-\vec{y}|$. To remedy this problem, consider a modified transformation (which we call MA′) of the form

$$\tilde{\phi}_{\vec{p}}(t,\vec{x})=\frac{1}{\sqrt{1-e^{\gamma+\gamma^*}}}(\phi_{\vec{p}}(t,\vec{x})-e^{\gamma}\phi^*_{-\vec{p}}(t,\vec{x}))$$
$$\equiv e^{i\vec{p}\cdot\vec{x}}\tilde{u}(p,t),\tag{4.1}$$

where

$$\tilde{u}(p,t)=\frac{1}{\sqrt{1-e^{\gamma+\gamma^*}}}(u(p,t)-e^{\gamma}u^*(p,t)).\tag{4.2}$$

Although this does seem to preserve translation invariance, it is far from obvious that Eq. (4.1) leads to a de Sitter invariant vacuum state for any complex γ [we again stick to ${\rm Re}(\gamma)<0$], but we will now see that this is indeed the case.

Let us start with $\gamma=-\infty$, so that $\tilde{\phi}=\phi$. Since the modes (2.17) are purely positive frequency on $\mathcal{I}^-$, we expect that if they define any de Sitter invariant vacuum at all, it should be the $|{\rm in}\rangle$ vacuum. To check this we calculate the Wightman two-point function

$$G^{\rm W}(t,\vec{x};t',\vec{x}')=\int d^2p\,\phi_{\vec{p}}(t,\vec{x})\phi^*_{\vec{p}}(t',\vec{x}')$$
$$=\frac{tt'}{8\pi\sinh\pi\mu}\int d^2p e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}$$
$$\times J_-(pt)J_+(pt')$$
$$=\frac{tt'}{4\sinh\pi\mu}\int_0^{\infty}dp\,pJ_0(p|\vec{x}-\vec{x}'|)$$
$$\times J_-(pt)J_+(pt').\tag{4.3}$$

It is straightforward but tedious to massage this integral [29] to obtain the result

$$G^{\rm W}=0,\quad P<-1,$$
$$=\frac{1}{4\pi\sinh\pi\mu}\frac{\cosh[\mu(\pi-\cos^{-1}(P))]}{\sin[\cos^{-1}(P)]},\quad -1<P<1,\tag{4.4}$$
$$=-\frac{i}{4\pi}{\rm sgn}(t-t')$$
$$\times\frac{\exp[-i\mu\,{\rm sgn}(t-t')\cosh^{-1}(P)]}{\sinh[\cosh^{-1}(P)]},\quad P>1,$$

with P given by Eq. (3.4). From Eq. (4.4) we find an expression for $G^{\rm C}=2\,{\rm Im}(G^{\rm W})$ which agrees with Eq. (3.5), and we find that the Hadamard function

$$G^{\mathrm{H}}=2\ \mathrm{Re}(G^{\mathrm{W}})=0, \quad P<-1,$$

$$=\frac{1}{2\pi\sinh\pi\mu}\times\frac{\cosh[\mu(\pi-\cos^{-1}(P))]}{\sin[\cos^{-1}(P)]}, \quad -1<P<1, \tag{4.5}$$

$$=-\frac{1}{2\pi}\frac{\sin[\mu\cosh^{-1}(P)]}{\sinh[\cosh^{-1}(P)]}, \quad P>1$$

agrees with Eq. (3.6) for $\gamma=-\infty$. This concludes the proof that the planar modes (2.17) define the $|\mathrm{in}\rangle$ vacuum.

Now consider arbitrary γ in Eq. (4.1). Since the commutator function is unaffected by this Bogolyubov transformation, we need only to calculate

$$G_\gamma^{\mathrm{H}}=2\ \mathrm{Re}\int d^2p\,\tilde{\phi}_{\vec{p}}(t,\vec{x})\tilde{\phi}_{\vec{p}}^*(t',\vec{x}'). \tag{4.6}$$

Using Eq. (4.1), we can write Eq. (4.6) as

$$G_\gamma^{\mathrm{H}}=\frac{1}{1-e^{\gamma+\gamma^*}}[(1+e^{\gamma+\gamma^*})G_{\mathrm{in}}^{\mathrm{H}}-4\ \mathrm{Re}(e^{\gamma^*}I)], \tag{4.7}$$

with $G_{\mathrm{in}}^{\mathrm{H}}$ given by Eq. (4.5) and

$$I=\frac{tt'}{8\pi\sinh\pi\mu}\int_0^\infty d^2p\,e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}J_-(pt)J_-(pt'). \tag{4.8}$$

It is again straightforward to check that Eq. (4.7) is equal to Eq. (3.6).

Finally we address the significance of the Euclidean vacuum from the point of view of the MA′ transform. Plugging $\gamma=-\pi\mu$ into Eq. (4.1), we find that the modes which give the Euclidean vacuum are

$$\phi_{\vec{p}}^{\mathrm{E}}(t,\vec{x})=\frac{te^{i\vec{p}\cdot\vec{x}}}{\sqrt{8\pi\sinh\pi\mu}}\frac{1}{\sqrt{1-e^{-2\pi\mu}}}\times[J_-(pt)-e^{-\pi\mu}J_+(pt)]. \tag{4.9}$$

Using the asymptotic expansion of the Bessel function we find that

$$\phi_{\vec{p}}^{\mathrm{E}}(t,\vec{x})\sim\frac{1}{2\pi}\sqrt{\frac{it}{2p}}e^{i\vec{p}\cdot\vec{x}-ipt} \tag{4.10}$$

near $t=\infty$. We see that precisely that linear combination (4.9) which gives the Euclidean vacuum is the one which is purely positive frequency near the horizon. This shows that the Euclidean vacuum is natural for cosmological purposes, when one might want to put boundary conditions on the scalar field on the past horizon of the region $\mathcal{O}^+$.

Before concluding our discussion of planar coordinates, we record here the momentum space Feynman propagator G^{F}, which will be used in the calculations below:

$$G_\gamma^{\mathrm{F}}(x,x')=\int d^2p\,e^{i\vec{p}\cdot(\vec{x}-\vec{x}')}\tilde{G}_\gamma^{\mathrm{F}}(t,t',p), \tag{4.11}$$

where we define the function

$$\tilde{G}_\gamma^{\mathrm{F}}(t,t',p)=\theta(t-t')\tilde{u}(p,t)\tilde{u}^*(p,t')+\theta(t'-t)\tilde{u}^*(p,t)\tilde{u}(p,t'). \tag{4.12}$$

V. dS/CFT IN GLOBAL COORDINATES

After the existence of an AdS/CFT correspondence was suggested by Maldacena [5], a precise prescription for calculating CFT correlation functions in terms of AdS data was soon developed [6,7]. In AdS/CFT, the gravity partition function, viewed as a functional of boundary data, serves as the generating functional of CFT correlators,

$$Z[\phi_0]=\left\langle \exp i\int_{\partial\mathcal{M}}\mathcal{O}\phi_0\right\rangle. \tag{5.1}$$

Recently the dS/CFT correspondence has been proposed by Strominger [1] and a recipe for calculating two-point CFT correlation functions has been suggested, but a precise dictionary between bulk and boundary correlation functions has not been given. There are at least two related reasons why adopting the AdS prescription is problematic.

As mentioned above, all of the modes in de Sitter space are normalizable, but in Lorentzian AdS, normalizable and non-normalizable modes play substantially different roles [23]. The former encode the states of the theory while the latter correspond to boundary conditions for fields and do not fluctuate. The AdS boundary conditions ensure, for example, that the on-shell action for a scalar field, which is a total derivative $S=\int dz\partial_z(z^{n-1}\phi\partial_z\phi)$, has only a contribution from the boundary $z=0$ and not from the horizon $z=\infty$. But in dS, there is no boundary condition [other than the trivial one $\phi(t,\vec{x})\equiv 0$] one could impose on ϕ at $\mathcal{I}^-$ in order to eliminate the contribution to the on-shell action from the horizon at $t=\infty$. This highlights the necessity of having two independent CFT operators for every bulk field ϕ, as opposed to the 1-1 correspondence familiar from AdS. (This argument applies in planar coordinates—of course in global coordinates one also expects two CFT operators, simply because there are two boundaries $\mathcal{I}^\pm$.) The fact that two CFT operators are associated with each bulk field has indeed been discussed in [1].

In evaluating the on-shell action in AdS space there are divergences which are easily regulated by prescribing boundary conditions not at $z=0$ but at $z=\epsilon$. A well-defined result is obtained after subtracting the power-law and logarithmic divergences as $\epsilon\rightarrow 0$. In dS a similarly regulated on-shell action for a scalar field is not infinite but does not converge—it has terms on $\mathcal{I}^-$ which behave like $e^{i/\epsilon}$ as $\epsilon\rightarrow 0$. In planar coordinates there are as mentioned in the previous paragraph also nonzero terms coming from the horizon which behave as e^{iT} as $T\rightarrow\infty$. We have been unable to find a regularization scheme which enables one to extract sensible results.

We proceed by recalling an alternative interpretation of the same AdS/CFT correlation functions which was developed in [30,10–13] and nicely proven in [14]. Giddings showed that the CFT correlators calculate the *S* matrix for scattering particles from the boundary of AdS into the bulk and back. In AdS/CFT, the boundary CFT has a time-like direction, and the positive and negative frequency components of a CFT operator $\mathcal{O}_\phi$ create and annihilate quanta of the associated bulk field ϕ.

We propose to adopt a suitable generalization of the construction of [14] to define CFT correlators for de Sitter space. Since the boundary conformal field theory is Euclidean, instead of having positive and negative frequency components of the operator we need the two operators $\mathcal{O}_+$, $\mathcal{O}_-$. In fact, in global coordinates it makes sense to think about four different operators $\mathcal{O}_+^{\rm in,out}$ and $\mathcal{O}_-^{\rm in,out}$. We interpret $\mathcal{O}_+^{\rm in}$ and $\mathcal{O}_-^{\rm in}$ as coupling respectively to positive and negative frequency quanta of the bulk field ϕ on $\mathcal{I}^-$. On $\mathcal{I}^+$ the pairing is reversed: $\mathcal{O}_+^{\rm out}$ couples to negative frequency quanta, and $\mathcal{O}_-^{\rm out}$ couples to positive frequency quanta. This convention ensures that operators $\mathcal{O}_\pm$ have conformal weight $h_\pm = 1 \pm i\mu$ regardless of whether they are in or out operators. Only two of the four operators are independent, and we will discuss below how to relate the out operators to the in operators perturbatively.

Concretely, our proposal is to define dS/CFT correlation functions in global coordinates by the prescription

$$\left\langle \prod_{i=1}^{m} \mathcal{O}_+^{\rm out}(\Omega_i) \prod_{j=1}^{n} \mathcal{O}_+^{\rm in}(\Omega_j') \right\rangle = \lim_{\substack{\tau_i\to+\infty \\ \tau_j'\to-\infty}} \int \left[\prod_{i=1}^{m} (\cosh\tau_i)^2 d^2\omega_i K^{\rm out *}(\Omega_i;x_i)\, i\overleftrightarrow{\partial}_{\tau_i} \right] \times G^{\rm F}(x_1,\ldots,x_m,x_1',\ldots,x_n') \times \left[\prod_{j=1}^{n} (\cosh\tau_j')^2 d^2\omega_j'\, i\overleftrightarrow{\partial}_{\tau_j'} K^{\rm in}(\Omega_j';x_j') \right], \quad (5.2)$$

where $G^{\rm F}$ is the bulk time-ordered Feynman correlation function and we use the notation $x=(\tau,\omega)$ [additional details of Eq. (5.2) will be clarified below].

The formula (5.2) only defines the two operators $\mathcal{O}_+^{\rm in,out}$, but it is straightforward to generalize the prescription to include the other two operators. Schematically, for every insertion of $\mathcal{O}_-^{\rm in}$ we include

$$\langle \cdots \mathcal{O}_-^{\rm in}(\Omega) \cdots \rangle = \lim_{\tau'\to-\infty} \int (\cosh\tau')^2 d^2\Omega' \times K^{\rm in *}(\Omega;x')\, i\overleftrightarrow{\partial}_{\tau'} G^{\rm F}(\ldots,x',\ldots), \quad (5.3)$$

while an insertion of $\mathcal{O}_-^{\rm out}$ involves

$$\langle \cdots \mathcal{O}_-^{\rm out}(\Omega) \cdots \rangle = \lim_{\tau'\to+\infty} \int (\cosh\tau')^2 d^2\Omega' \times G^{\rm F}(\ldots,x',\ldots)\, i\overleftrightarrow{\partial}_{\tau'} K^{\rm out}(\Omega;x'). \quad (5.4)$$

The ordering of the operators inside these correlation functions is irrelevant, except for possible contact terms, which can be computed explicitly (as we will show below).

The motivation for the proposal (5.2) comes from studying *S*-matrix elements in dS_3. In the next subsection we will derive an LSZ-like formula for the *S* matrix and show that it can be written in terms of the correlation functions (5.2) as

$$S[\{f_i\};\{g_j\}] = \int \left[\prod_{i=1}^{m} \frac{d^2\Omega_i}{\sqrt{Z}} f_i^*(\Omega_i) \right] \times \left[\prod_{j=1}^{n} \frac{d^2\Omega_j'}{\sqrt{Z}} g_j(\Omega_j') \right] \times \left\langle \prod_{i=1}^{m} \mathcal{O}_+^{\rm out}(\Omega_i) \prod_{j=1}^{n} \mathcal{O}_+^{\rm in}(\Omega_j') \right\rangle, \quad (5.5)$$

where f_i and g_j are smooth functions on the sphere, and the left-hand side is the *S*-matrix element for *n* incoming wave packets with envelopes f_i and *m* outgoing wave packets with envelopes g_j. The factor *Z* is a wave function renormalization which one could calculate perturbatively.

A. Motivation: The *S* matrix

Following standard arguments [31], we consider the interaction of some wave packets which are widely separated in the far past and in the far future, so that the full interacting field $\Phi(x)$ asymptotes to free fields,

$$\lim_{\tau\to-\infty} \Phi(x) = \sqrt{Z}\phi^{\rm in}(x), \qquad \lim_{\tau\to+\infty} \Phi(x) = \sqrt{Z}\Phi^{\rm out}(x). \quad (5.6)$$

Here we allow for a wave function renormalization *Z*, and the canonically normalized free fields $\Phi^{\rm in,out}$ are expanded in terms of operators $a^{\rm in,out}$ as

$$\Phi^{\rm in,out} = \sum_{lm} \phi_{lm} a_{lm}^{\rm in,out} + \phi_{lm}^* a_{lm}^{\rm in,out\dagger}, \quad a_{lm}^{\rm in,out}|0\rangle = 0. \quad (5.7)$$

Note that we are taking the in and out vacua to be the same, as is appropriate for dS_3 [8]. The role of the choice of vacuum will be discussed below. The condition (5.6) holds weakly (i.e., it is not an operator identity but is valid inside matrix elements). The operators $a^{\rm in,out}$ are recovered from the free fields $\Phi^{\rm in,out}$ by the standard formula

$$a_{lm}^{\rm in,out} = \int (\cosh\tau)^2 d^2\Omega\, \phi_{lm}^*(\tau,\Omega)\, i\overleftrightarrow{\partial}_\tau \Phi^{\rm in,out}(\tau,\Omega). \quad (5.8)$$

Since Φ^{in} and Φ^{out} both satisfy the free wave equation, these operators are independent of τ.

Given a smooth function $g(\Omega)$ on $\mathcal{I}^-$, we can use the bulk-boundary propagator K^{in} of Sec. II A to construct a solution ϕ_g^{in} of the wave equation which represents an incoming wave packet with envelope g, as in Eq. (2.16). This wave packet corresponds to the state $\alpha_g^{\text{in}}|0\rangle$, where

$$\alpha_f^{\text{in,out}} \equiv \int (\cosh\tau)^2 d^2\Omega\, \phi_f^{\text{in,out}*}(\tau,\Omega) \times i\overleftrightarrow{\partial}_\tau \Phi^{\text{in,out}}(\tau,\Omega). \tag{5.9}$$

Similarly, an outgoing wave packet with envelope f at $\mathcal{I}^+$ is constructed using K^{out}, and corresponds to the state $\langle 0|\alpha_f^{\text{out}}$. The S-matrix element for n incoming wave packets $\{g_j\}$ and m outgoing wave packets $\{f_i\}$ is defined by

$$S[\{f_i\};\{g_j\}] = \langle 0|\prod_{i=1}^m \alpha_{f_i}^{\text{out}} \prod_{j=1}^n \alpha_{g_j}^{\text{in}\dagger}|0\rangle. \tag{5.10}$$

Now using the definitions (5.9), (5.10) and the asymptotic condition (5.6), it is straightforward to derive a formula for the S matrix:

$$S[\{f_i\};\{g_j\}] = \lim_{\substack{\tau_i\to+\infty\\ \tau_j'\to-\infty}} \int \left[\prod_{i=1}^m (\cosh\tau_i)^2 \frac{d^2\Omega_i}{\sqrt{Z}} \times \phi_{f_i}^{\text{out}*}(x_i)\, i\overleftrightarrow{\partial}_{\tau_i}\right] \times \langle 0|T\prod_{i=1}^m \Phi(x_i)\prod_{j=1}^n \Phi(x_j')|0\rangle \times \left[\prod_{j=1}^n (\cosh\tau_j')^2 \frac{d^2\Omega_j'}{\sqrt{Z}}\, i\overleftrightarrow{\partial}_{\tau_j'} \phi_{g_j}^{\text{in}}(x_j')\right]. \tag{5.11}$$

Note that the derivative operators do not hit the factors of $(\cosh\tau)^2$ in the measure, as in Eq. (5.9). Also, the time-ordering symbol inside the bulk correlation function can be interpreted as defining the order in which the τ coordinates are taken to infinity. One should evaluate the quantity at a fixed $\tau_1 > \cdots \tau_m > \tau_1' > \cdots > \tau_n'$ and then take the limits preserving that ordering. In particular, one need not worry about delta function contributions coming from when the τ derivatives hit the time-ordering symbol. Note that the vacuum $|0\rangle$ in Eq. (5.11) can be any of the vacuum states discussed in the previous sections.[9] We will see by explicit calculation how the S-matrix elements, and hence the CFT correlators, depend on this choice of vacuum.

[9]In fact, there is no reason to necessarily take the incoming and outgoing vacua to be the same, although we will not pursue this possibility here.

So far we have only used the operators $\mathcal{O}_+^{\text{in,out}}$, but it is clear how to introduce the other two. In the S matrix (5.10) we can also include operators like α^{in} and $\alpha^{\text{out}\dagger}$, which lead straightforwardly to the prescriptions (5.3), (5.4) for insertions of $\mathcal{O}_-^{\text{in}}$ and $\mathcal{O}_-^{\text{out}}$, respectively. The possibility of including these operators may seem unfamiliar since normally one can use only $a^{\text{in}\dagger}$ and not a^{in} for constructing initial states since the latter annihilates $|0\rangle$. But we can include them here since we will be interested in the γ dependence of the CFT correlation functions, and a^{in} only annihilates $|\gamma\rangle$ for $\gamma = -\infty$, the $|\text{in}\rangle$ vacuum.

B. Relation between in and out operators

Using Green's theorem and the fact that $K^{\text{out}}(\Omega;x)$ satisfies the equation of motion $(\nabla_x^2 - m^2)K^{\text{out}} = 0$ immediately gives the formula

$$\begin{aligned} -i\int \sqrt{-g}\,dx' K^{\text{out}}(\Omega;x')(\nabla_{x'}^2 - m^2) &\times G^{\text{F}}(\dots,x',\dots) \\ = \left(\lim_{\tau'\to+\infty} - \lim_{\tau'\to-\infty}\right)\int d^2\Omega'\, i(\cosh\tau')^2 &\times G^{\text{F}}(\dots,x',\dots)\overleftrightarrow{\partial}_{\tau'} K^{\text{out}}(\Omega;x'). \end{aligned} \tag{5.12}$$

Now the first term on the right-hand side looks like an insertion of $\mathcal{O}_-^{\text{out}}$, as in Eq. (5.4), while the second term on the right-hand side can be made to look like an insertion of $\mathcal{O}_+^{\text{in}}$ by recalling the relation (2.13). This leads to the identity

$$\begin{aligned} \langle\cdots\mathcal{O}_-^{\text{out}}(\Omega)\cdots\rangle = -\mu\sinh\pi\mu \int d\Omega'\,\Delta_-(\Omega,\Omega_A') &\times \langle\cdots\mathcal{O}_+^{\text{in}}(\Omega')\cdots\rangle \\ &- i\int \sqrt{-g}\,dx' K^{\text{out}}(\Omega;x') \\ &\times (\nabla_{x'}^2 - m^2)G^{\text{F}}(\dots,x',\dots). \end{aligned} \tag{5.13}$$

Of course a similar formula relates $\mathcal{O}_-^{\text{in}}$ and $\mathcal{O}_+^{\text{out}}$. For two-point functions in the free theory, it is not hard to see that the second line of Eq. (5.13) vanishes, so that one obtains the weak operator identities

$$\mathcal{O}_\pm^{\text{out}}(\Omega) = -\mu\sinh\pi\mu\int d\Omega'\,\Delta_\pm(\Omega,\Omega_A')\mathcal{O}_\mp^{\text{in}}(\Omega'). \tag{5.14}$$

This relation receives perturbative corrections which can in principle be derived from the identity (5.13).

C. CFT two-point functions

We now show that the proposal (5.2) reproduces the two-point functions of [8] in an arbitrary vacuum. This calculation is trivial in momentum space, so we start by Fourier transforming Eq. (5.2) to obtain

$$\left\langle \prod_{i=1}^{m} \mathcal{O}^{\rm in}_{-l_i m_i} \prod_{j=1}^{n} \mathcal{O}^{\rm in}_{+l'_j m'_j} \right\rangle = \lim_{\substack{\tau_i\to-\infty \\ \tau'_j\to-\infty}} \left[\prod_{i=1}^{n} k^*_{l_i}(\tau_i) i(\cosh\tau_i)^2 \overleftrightarrow{\partial}_{\tau_i} \right] \times \tilde{G}^{\rm F}_{\{l_i m_i\},\{l'_j, m'_j\}}(\{\tau_i\},\{\tau'_j\}) \times \left[\prod_{j=1}^{m} i(\cosh\tau'_j)^2 \overleftrightarrow{\partial}_{\tau'_j} k_{l'_j}(\tau'_j) \right], \quad (5.15)$$

where we have defined

$$k_l(\tau) \equiv e^{-i\theta_l}\sqrt{\frac{\mu}{2}}\, y_l(\tau). \quad (5.16)$$

This is essentially just $K^{\rm in}$, but with the spherical harmonics stripped off already.

Since $k_l(\tau)$ is just $y_l(\tau)$ up to a factor, it is trivial to use the momentum space representation (3.7) and the orthogonality of the modes (2.8) to obtain

$$\langle \gamma | \mathcal{O}^{\rm in}_{-lm} \mathcal{O}^{\rm in}_{-l'm'} | \gamma \rangle = -\frac{e^{\gamma^*}}{1-e^{\gamma+\gamma^*}} e^{2i\theta_l} \frac{\mu}{2}, \quad \text{etc.} \quad (5.17)$$

Fourier transforming back to position space gives

$$\langle \gamma | \mathcal{O}^{\rm in}_{-}(\Omega) \mathcal{O}^{\rm in}_{-}(\Omega') | \gamma \rangle = \frac{\mu^2}{2} \sinh \pi\mu \frac{e^{\gamma^*}}{1-e^{\gamma+\gamma^*}} \times \Delta_-(\Omega,\Omega'),$$

$$\langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega) \mathcal{O}^{\rm in}_{+}(\Omega') | \gamma \rangle = \frac{\mu^2}{2} \sinh \pi\mu \frac{e^{\gamma}}{1-e^{\gamma+\gamma^*}} \times \Delta_+(\Omega,\Omega'),$$

$$\langle \gamma | \mathcal{O}^{\rm in}_{-}(\Omega) \mathcal{O}^{\rm in}_{+}(\Omega') | \gamma \rangle = \frac{\mu}{2} \frac{1}{1-e^{\gamma+\gamma^*}} \delta^2(\Omega,\Omega'),$$

$$\langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega) \mathcal{O}^{\rm in}_{-}(\Omega') | \gamma \rangle = \frac{\mu}{2} \frac{e^{\gamma+\gamma^*}}{1-e^{\gamma+\gamma^*}} \delta^2(\Omega,\Omega'), \quad (5.18)$$

in agreement with the results of [8] (after translating from our γ conventions to their α conventions).

D. CFT three-point function

Of course as far as the two-point functions (5.18) are concerned, one could eliminate the γ dependence by rescaling the operators $\mathcal{O}^{\rm in,out}_{\pm}$. In this section we outline the calculation of a CFT three-point function in the presence of a ϕ^3 interaction in the bulk, and prove that an invariant ratio of correlation functions depends nontrivially on γ. This provides evidence that these $|\gamma\rangle$ vacua are marginal deformations of the CFT, as opposed to simply field rescalings. The calculation appears more difficult than the corresponding calculation in AdS/CFT,[10] but fortunately we will be able to exploit the simple behavior of the global coordinate modes under the antipodal map to extract the essential features of the result. The invariant ratio we will calculate is

$$R(\gamma) \equiv \frac{\langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega_1) \mathcal{O}^{\rm in}_{+}(\Omega_2) \mathcal{O}^{\rm in}_{+}(\Omega_3) | \gamma \rangle^2}{\langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega_1) \mathcal{O}^{\rm in}_{+}(\Omega_2) | \gamma \rangle \langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega_1) \mathcal{O}^{\rm in}_{+}(\Omega_3) | \gamma \rangle \langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega_2) \mathcal{O}^{\rm in}_{+}(\Omega_3) | \gamma \rangle}. \quad (5.19)$$

Since our calculation will not be able to determine the overall (γ-independent) constant in R, we omit overall constants throughout this calculation. The prescription (5.2) amounts to extracting the coefficient of $e^{h_+(\tau_1+\tau_2+\tau_3)}$ as all three points approach $\mathcal{I}^-$. That is,

$$\lim_{\tau_i\to-\infty} G^{\rm F}_{\gamma}(x_1,x_2,x_3) \sim e^{h_+(\tau_1+\tau_2+\tau_3)} \langle \gamma | \mathcal{O}^{\rm in}_{+}(\Omega_1) \mathcal{O}^{\rm in}_{+}(\Omega_2) \mathcal{O}^{\rm in}_{+}(\Omega_3) | \gamma \rangle + \cdots. \quad (5.20)$$

Diffeomorphism invariance of $G^{\rm F}_{\gamma}(x_1,x_2,x_3)$ ensures that the CFT three-point function read off in this manner will be conformally invariant.

[10]In planar coordinates, the technical difficulty arises because the three-point function involves integrals like $\int d^2y K(\vec{x}_1;\vec{y}) K(\vec{x}_2;\vec{y}) K(\vec{x}_3;\vec{y})$, but the θ function in the bulk boundary propagator K (2.19) makes this integral difficult to manipulate. In particular, the clever AdS tricks of [32] do not seem to work.

At the tree level in perturbation theory we have

$$G^{\rm F}_\gamma(x_1,x_2,x_3)=\int \sqrt{-g}dx G^{\rm F}_\gamma(x,x_1)G^{\rm F}_\gamma(x,x_2)\times G^{\rm F}_\gamma(x,x_3). \tag{5.21}$$

Since we are interested in the limit $\tau_i\to-\infty$, it is safe to replace the time-ordered two-point functions in Eq. (5.21) by the Wightman function (it is not hard to check carefully that the difference of the integrals goes to zero in the limit we are interested in). Then we use the identity (3.2), but note that the second and third terms in Eq. (3.2) behave as $e^{h_-\tau'}$ near $\mathcal{I}^-$ and therefore do not contribute to Eq. (5.19). Keeping only the terms which behave like $e^{h_+(\tau_1+\tau_2+\tau_3)}$ gives

$$(1-e^{\gamma+\gamma^*})^{-3}\int \sqrt{-g}dx\prod_{i=1}^{3}\left[G^{\rm W}(x,x_i)-e^{\gamma}G^{\rm W}(x_A,x_i)\right]. \tag{5.22}$$

The eight terms in Eq. (5.22) can easily be combined by noting that in every integral with two or three $x_{\rm A}$'s we can make the change of variables $x\to x_{\rm A}$ to end up with only one $x_{\rm A}$ or none. Therefore Eq. (5.22) is equal to

$$(1-e^{\gamma+\gamma^*})^{-3}[(1-e^{3\gamma})G-e^{\gamma}(1-e^{\gamma})(G_1+G_2+G_3)], \tag{5.23}$$

where

$$G\equiv\int \sqrt{-g}dxG^{\rm W}(x,x_1)G^{\rm W}(x,x_2)G^{\rm W}(x,x_3) \tag{5.24}$$

and

$$G_1\equiv\int \sqrt{-g}dxG^{\rm W}(x_{\rm A},x_1)G^{\rm W}(x,x_2)G^{\rm W}(x,x_3),\quad \text{etc.} \tag{5.25}$$

In fact it is safe to replace $G^{\rm W}$ by the time-ordered product $G^{\rm F}$ in Eqs. (5.24) and (5.25) since we are only interested in the limit $\tau_i\to-\infty$. In any case, diffeomorphism invariance of the integrals (5.24) and (5.25) implies that coefficients of $e^{h_+(\tau_1+\tau_2+\tau_3)}$ must be proportional to the conformally invariant three-point function Δ_{+++} for a field of weight h_+. We have not determined the constants of proportionality, but since there is no γ dependence in the remaining integrals (5.24) and (5.25), the γ dependence of Eq. (5.22) must be of the form

$$\langle\gamma|\mathcal{O}^{\rm in}_+(\Omega_1)\mathcal{O}^{\rm in}_+(\Omega_2)\mathcal{O}^{\rm in}_+(\Omega_3)|\gamma\rangle \sim(1-e^{\gamma+\gamma^*})^{-3}[x(1-e^{3\gamma})-ye^{\gamma}(1-e^{\gamma})]\Delta_{+++}, \tag{5.26}$$

where x and y are undetermined nonzero constants. We conclude that the invariant ratio (5.19) is

$$R(\gamma)\sim e^{-3\gamma}(1-e^{\gamma+\gamma^*})^{-3}[x(1-e^{3\gamma})-ye^{\gamma}(1-e^{\gamma})]^2. \tag{5.27}$$

Although we have not determined x or y, it is clear that no choice renders $R(\gamma)$ independent of γ. Therefore we conclude that the γ dependence of the CFT correlation functions cannot be absorbed into a rescaling of the operators $\mathcal{O}$.

VI. dS/CFT IN PLANAR COORDINATES

In planar coordinates, we propose to define dS/CFT correlation functions by the rule

$$\left\langle\prod_{i=1}^{m}\mathcal{O}_-(\vec{x}_i)\prod_{j=1}^{n}\mathcal{O}_+(\vec{y}_j)\right\rangle =\lim_{t_i,t'_j\to0}\int\left[\prod_{i=1}^{m}\frac{d^2x_i}{t_i}K^*(\vec{x}_i;x_i)i\overleftrightarrow{\partial}_{t_i}\right]\times G^{\rm F}(x'_1,\dots,x'_m;y'_1,\dots,y'_n)\times\left[\prod_{j=1}^{n}\frac{d^2y'_j}{t'_j}i\overleftrightarrow{\partial}_{t'_j}K(\vec{y}'_j;y'_j)\right], \tag{6.1}$$

with the notation $x=(t,\vec{x})$ and $y=(t',\vec{y})$. Again the ordering of the operators is irrelevant except for contact terms, which can be computed by ordering the x'_i and y'_j in parallel with the corresponding $\vec{x}_i$ and $\vec{y}_j$. In the next subsection we motivate this definition by analyzing the S vector [2].

A. Motivation: The *S* vector

In planar coordinates covering $\mathcal{O}^-$ it does not make sense to speak of asymptotic out states since the horizon is located at a finite affine distance from any point in the bulk of $\mathcal{O}^-$. (In our formalism, this problem manifests itself through the lack of a "bulk-horizon" propagator which one could use to propagate wave packets from the horizon.) Therefore it has been proposed [2] (see also [15]) that the natural metaobservable is not the S matrix but an S vector, where a unique state $\langle U|$ is generated by some unknown mechanism on the horizon, and the only calculable quantities are $\langle U|a\rangle$, for states $|a\rangle$ on the boundary (which we take to be $\mathcal{I}^-$). This is the point of view we will adopt, although for simplicity we will only consider the case when $\langle U|$ is one of the de Sitter invariant vacuum states $|\gamma\rangle$.

We define the S vector

$$S[\{f_i\};\{g_j\}]=\langle0|\prod_{i=1}^{m}\alpha^{\rm in}_{f_i}\prod_{j=1}^{n}\alpha^{\rm in\dagger}_{g_j}|0\rangle, \tag{6.2}$$

where

$$\alpha^{\rm in}_f=\frac{i}{t}\int d^2x\,\phi_f(t,\vec{x})\overleftrightarrow{\partial}_t\Phi^{\rm in}(t,\vec{x}), \tag{6.3}$$

with ϕ_f defined in Eq. (2.22). We have kept the superscript "in" in these formulas to compare with the previous section, but since here there is no ''out,'' they will henceforth be dropped. Note that the operator α_f in Eq. (6.2) annihilates a wave packet with envelope f at $\mathcal{I}^-$. This makes sense because a general de Sitter invariant vacuum state $|0\rangle$ which we might choose to use in Eq. (6.2) actually contains an infinite number of particles on $\mathcal{I}^-$. We form the initial state by adding the wave packets g_j and deleting the wave packets f_i from this state.

Repeating the LSZ analysis of the previous section, it is easy to write the S vector in the form

$$S[\{f_i\};\{g_j\}]=\int\left[\prod_{i=1}^{m}\frac{d^2x_i}{\sqrt{Z}}f_i^{IL}(\vec{x}_i)\right]\times\left[\prod_{j=1}^{n}\frac{d^2y_j}{\sqrt{Z}}g_j(\vec{y}_j)\right]\times\left\langle\prod_{i=1}^{m}\mathcal{O}_-(\vec{x}_i)\prod_{j=1}^{n}\mathcal{O}_+(\vec{y}_j)\right\rangle, \tag{6.4}$$

with the CFT correlator on the right hand side given by Eq. (6.1).

B. CFT two-point functions

In momentum space, the CFT correlation function (6.1) is simply

$$\left\langle\prod_{i=1}^{m}\mathcal{O}_-(\vec{p}_i)\prod_{j=1}^{n}\mathcal{O}_+(\vec{q}_j)\right\rangle=\lim_{t_i,t_j'\to 0}\left[\prod_{i=1}^{m}k^*(p_i,t_i)\frac{i}{t_i}\overleftrightarrow{\partial}_{t_i}\right]\times G^{\mathrm{F}}(\{t_i,\vec{p}_i\},\{t_j',\vec{q}_j\})\times\left[\prod_{j=1}^{n}\frac{i}{t_j'}\overleftrightarrow{\partial}_{t_j'}k(q_j,t_j')\right], \tag{6.5}$$

with $k(p,t)\equiv 2\pi z(p)u(p,t)$, recalling Eqs. (2.19) and (2.20).

Using Eq. (4.12) and the orthogonality (2.18) of the modes, we find immediately

$$\langle\gamma|\mathcal{O}_-(\vec{x})\mathcal{O}_-(\vec{y})|\gamma\rangle=-\frac{e^{\gamma^*}}{1-e^{\gamma+\gamma^*}}\int d^2p\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}(2\pi z^*(p))^2=-16i(\pi\mu)^2\frac{e^{\gamma^*}}{1-e^{\gamma+\gamma^*}}\frac{1}{|\vec{x}-\vec{y}|^{2h_-}} \tag{6.6}$$

and

$$\langle\gamma|\mathcal{O}_-(\vec{x})\mathcal{O}_+(\vec{y})|\gamma\rangle=\frac{1}{1-e^{\gamma+\gamma^*}}\int d^2p\,e^{i\vec{p}\cdot(\vec{x}-\vec{y})}\times|2\pi z(p)|^2=\frac{1}{1-e^{\gamma+\gamma^*}}2\mu(2\pi)^4\delta(\vec{x}-\vec{y}). \tag{6.7}$$

VII. SUMMARY AND DISCUSSION

The purpose of this paper has primarily been to define a procedure for calculating CFT correlation functions from bulk n-point functions in dS_3. Although our proposal is modeled on a similar procedure from AdS/CFT, we have highlighted some of the important differences between dS and AdS which make naive extrapolation of AdS results impossible. These differences include the fact that in dS one inevitably has two CFT operators for every bulk field ϕ (since there is no natural boundary condition one could impose to eliminate the second operator), as well as the fact that a scalar field in de Sitter space has a whole family of different vacuum states, none of which is the one obtained from AdS by analytic continuation.

We have also shown that these de Sitter invariant vacuum states arise naturally in coordinates covering only half of de Sitter space, where the Euclidean vacuum plays the special role of having no particles on the horizon. Finally, we have sketched the calculation of a CFT three-point function and shown that an invariant ratio (5.19) of correlation functions depends nontrivially on the choice of vacuum γ. This shows that the γ dependence of the CFT correlation functions cannot be eliminated by rescaling the operators. However, it leaves open the intriguing possibility that the correlation functions may be related by a γ-dependent *nonlocal* field redefinition of $\mathcal{O}_\pm^{\mathrm{in,out}}$. This is easily seen to be true for the two-point functions (5.18), and it would be interesting to see whether this is a general feature.

Our S-matrix and S-vector proposals answer the question of what these CFT correlation functions of [2,1,8] are. Unfortunately, we have not answered the interesting and pressing question of how to interpret these quantities, which have been called ''metaobservables'' [2] since no single observer in de Sitter space can access more than a single point on $\mathcal{I}^+$. Also, in this formulation of the dS/CFT, the CFT lives on a Cauchy surface at infinite distance, rather than a boundary. It might be more satisfactory, from a holographic point of view, to have a formulation in which the CFT lives on the horizon [33].

ACKNOWLEDGMENTS

We have benefited greatly from discussions with R. Bousso, I. Klebanov, A. Maloney, I. Savonije, K. Skenderis, E. Verlinde, and H. Verlinde. We are especially grateful to A. Strominger for many useful discussions and to the authors of [8] for early drafts. M.S. is supported by DOE grant DE-FG02-91ER40671, and A.V. is supported by DE-FG02-91ER40655.

[1] A. Strominger, J. High Energy Phys. **10**, 034 (2001).
[2] E. Witten, ''Quantum gravity in de Sitter space,'' hep-th/0106109.
[3] F. L. Lin and Y. S. Wu, Phys. Lett. B **453**, 222 (1999); D. Klemm, Nucl. Phys. **B625**, 295 (2002); J. Bros, H. Epstein, and U. Moschella, Phys. Rev. D (to be published), hep-th/0107091; S. Cacciatori and D. Klemm, Class. Quantum Grav. **19**, 579 (2002); B. McInnes, ''Exploring the similarities of the dS/CFT and AdS/CFT correspondences,'' hep-th/0110062; V. Balasubramanian, J. de Boer, and D. Minic, ''Mass, entropy and holography in asymptotically de Sitter spaces,'' hep-th/0110108; B. G. Carneiro da Cunha, Phys. Rev. D (to be published), hep-th/0110169; R. G. Cai, Phys. Lett. B **525**, 331 (2002); M. H. Dehghani, Phys. Rev. D (to be published), hep-th/0112002; B. McInnes, ''The dS/CFT correspondence and the big smash,'' hep-th/0112066; Y. S. Myung, ''Dynamic dS/CFT correspondence using the brane cosmology,'' hep-th/0112140; S. Nojiri and S. D. Odintsov, ''Wilson loop and dS/CFT correspondence,'' hep-th/0112152.
[4] J. Maldacena and A. Strominger, J. High Energy Phys. **02**, 014 (1998); R. Bousso, Phys. Rev. D **58**, 083511 (1998); M. I. Park, Phys. Lett. B **440**, 275 (1998); C. M. Hull, J. High Energy Phys. **07**, 021 (1998); M. Banados, T. Brotz, and M. E. Ortiz, Phys. Rev. D **59**, 046002 (1999); W. T. Kim, *ibid.* **59**, 047503 (1999); R. Bousso, *ibid.* **60**, 063503 (1999); S. Hawking, J. Maldacena, and A. Strominger, J. High Energy Phys. **05**, 001 (2001); R. Bousso, *ibid.* **04**, 035 (2001); M. K. Parikh and S. N. Solodukhin, Phys. Lett. B **503**, 384 (2001); A. Volovich, ''Discreteness in de Sitter space and quantization of Kaehler manifolds,'' hep-th/0101176; A. Chamblin and N. D. Lambert, Phys. Lett. B **508**, 369 (2001); V. Balasubramanian, P. Horava, and D. Minic, J. High Energy Phys. **05**, 043 (2001); M. Li, ''Matrix model for de Sitter,'' hep-th/0106184; Y. H. Gao, ''Symmetries, matrices, and de Sitter gravity,'' hep-th/0107067; S. Nojiri and S. D. Odintsov, J. High Energy Phys. **12**, 033 (2001); E. Halyo, ''De Sitter entropy and strings,'' hep-th/0107169; J. W. Moffat, ''M-theory and de Sitter space,'' hep-th/0107183; R. Kallosh, ''N=2 supersymmetry and de Sitter space,'' hep-th/0109168; C. M. Hull, J. High Energy Phys. **11**, 012 (2001); **11**, 061 (2001); S. Nojiri and S. D. Odintsov, Phys. Lett. B **523**, 165 (2001); R. Kallosh, A. D. Linde, S. Prokushkin, and M. Shmakova, ''Gauged supergravities, de Sitter space and cosmology,'' hep-th/0110089; T. Shiromizu, Phys. Rev. D **60**, 064019 (1999); R. G. Cai, Y. S. Myung, and Y. Z. Zhang, Phys. Rev. D (to be published), hep-th/0110234; U. H. Danielsson, ''A black hole hologram in de Sitter space,'' hep-th/0110265; S. Ogushi, Mod. Phys. Lett. A **17**, 51 (2002); A. M. Ghezelbash and R. B. Mann, J. High Energy Phys. **01**, 005 (2002); G. W. Gibbons and C. M. Hull, ''de Sitter space from warped supergravity solutions,'' hep-th/0111072; D. Youm, ''The Cardy-Verlinde formula and asymptotically de Sitter brane universe,'' hep-th/0111276; M. Cvetic, S. Nojiri, and S. D. Odintsov, ''Black Hole Thermodynamics and Negative Entropy in de Sitter and Anti-de Sitter Einstein-Gauss-Bonnet gravity,'' hep-th/0112045; P. Berglund, T. Hubsch, and D. Minic, ''de Sitter spacetimes from warped compactifications of IIB string theory,'' hep-th/0112079.
[5] J. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1998)].
[6] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. B **428**, 105 (1998).
[7] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998).
[8] R. Bousso, A. Maloney, and A. Strominger, ''Conformal Vacua and Entropy in de Sitter Space,'' hep-th/0112218.
[9] T. Banks and W. Fischler, ''M-Theory observables for cosmological space-times,'' hep-th/0102077.
[10] J. Polchinski, ''S-matrices from AdS spacetime,'' hep-th/9901076.
[11] L. Susskind, ''Holography in the flat space limit,'' hep-th/9901079.
[12] V. Balasubramanian, S. B. Giddings, and A. E. Lawrence, J. High Energy Phys. **03**, 001 (1999).
[13] I. Y. Aref'eva, ''On the holographic S-matrix,'' hep-th/9902106.
[14] S. B. Giddings, Phys. Rev. Lett. **83**, 2707 (1999).
[15] S. Hellerman, N. Kaloper, and L. Susskind, J. High Energy Phys. **06**, 003 (2001).
[16] W. Fischler, A. Kashani-Poor, R. McNees, and S. Paban, J. High Energy Phys. **07**, 003 (2001).
[17] M. Spradlin, A. Strominger, and A. Volovich, ''Les Houches lectures on de Sitter space,'' hep-th/0110007.
[18] A. Strominger, J. High Energy Phys. **11**, 049 (2001).
[19] A. J. Tolley and N. Turok, ''Quantization of the massless minimally coupled scalar field and the dS/CFT correspondence,'' hep-th/0108119.
[20] P. Breitenlohner and D. Z. Freedman, Ann. Phys. (N.Y.) **144**, 249 (1982).
[21] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999).
[22] P. Minces and V. O. Rivelles, J. High Energy Phys. **12**, 010 (2001).
[23] V. Balasubramanian, P. Kraus, and A. E. Lawrence, Phys. Rev. D **59**, 046003 (1999).
[24] N. A. Chernikov and E. A. Tagirov, Ann. Inst. Henri Poincare, Sect. A **9**, 109 (1968).
[25] E. A. Tagirov, Ann. Phys. (N.Y.) **76**, 561 (1973).
[26] B. Allen, Phys. Rev. D **32**, 3136 (1985).
[27] E. Mottola, Phys. Rev. D **31**, 754 (1985).
[28] S. A. Fulling, *Aspects of Quantum Field Theory in Curved Space-Time* (Cambridge University Press, Cambridge, England, 1989).
[29] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 2000).
[30] V. Balasubramanian, P. Kraus, A. E. Lawrence, and S. P. Trivedi, Phys. Rev. D **59**, 104021 (1999).
[31] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Fields* (McGraw-Hill, New York, 1965).
[32] D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, Nucl. Phys. **B546**, 96 (1999).
[33] I. Sachs and S. N. Solodukhin, Phys. Rev. D **64**, 124023 (2001).